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# Finite self dual groups ☆

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ABSTRACT

A group $G$ is $s$-self dual if every subgroup of $G$ is isomorphic to a quotient of $G$. It is $q$-self dual if every quotient of $G$ is isomorphic to a subgroup of $G$. It is self dual if it is both $s$-self dual and $q$-self dual. In this paper, the structure of finite self dual groups is determined.



## 1. Introduction

A group $G$ is *$s$-self dual* if every subgroup of $G$ is isomorphic to a quotient of $G$. It is *$q$-self dual* if every quotient of $G$ is isomorphic to a subgroup of $G$. It is *self dual* if it is both $s$-self dual and $q$-self dual. The study of finite self dual groups was initiated by Spencer in [6]. He obtained the following results:

**Theorem 1.1.** *A finite group $G$ is self dual if and only if $G$ is nilpotent and all Sylow subgroups of $G$ are self dual.*

**Theorem 1.2.** *If $P$ is a finite $p$-group, then each subgroup of $P$ is self dual if and only if $P$ is abelian or $P = H \times K$, where $H$ is the extraspecial group of order $p^3$ and exponent $p$, and $K$ is elementary abelian.*

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By Theorem 1.1, to study finite self dual groups, we only need to study finite self dual $p$-groups. A natural question is: what is the structure of finite self dual $p$-groups? In fact, classifying such groups is Problem 706 proposed by Berkovich and Janko in [1].

In this paper, the structure of finite $s$-self dual groups is determined, and as a by-product, the structure of finite self dual groups is also determined. Naturally we propose the following

**Question.** What is the structure of $q$-self dual groups?

## 2. Preliminaries

Let $G$ be a finite $p$-group. We use $c(G)$ and $d(G)$ to denote the nilpotency class of $G$ and the minimal number of generators respectively. We define $\Omega_1(G) = \langle a \in G \mid a^p = 1\rangle$ and $\mho_1(G) = \langle a^p \mid a \in G\rangle$. Let

$$G > G_2 > \cdots > G_{c+1} = 1$$

denote the lower central series of $G$, where $c = c(G)$. We use $C_n$ and $C_n^m$ to denote the cyclic group and the direct product of $m$ cyclic groups of order $n$, respectively.

If $G$ is a finite group, then $G$ is *minimal non-abelian* if $G$ is non-abelian, but every proper subgroup of $G$ is abelian.

We use $M_p(m,n)$ to denote groups $\langle a, b \mid a^{p^m} = b^{p^n} = 1,\ a^b = a^{1+p^{m-1}}\rangle$, where $m \geqslant 2$. We use $M_p(m,n,1)$ to denote groups $\langle a,b,c \mid a^{p^m} = b^{p^n} = c^p = 1,\ [a,b] = c,\ [c,a] = [c,b] = 1\rangle$, where $m \geqslant n$, and if $p = 2$, then $m + n \geqslant 3$. For other notation and terminology the reader is referred to [3].

**Theorem 2.1.** *(See [5].) If $G$ is a minimal non-abelian $p$-group, then $G$ is $Q_8$, $M_p(m,n)$, or $M_p(m,n,1)$.*

**Theorem 2.2.** *(See [8, Lemma 2.2].) If $G$ is a finite $p$-group, then the following conditions are equivalent*:

(1) *$G$ is a minimal non-abelian $p$-group*;
(2) *$d(G) = 2$ and $|G'| = p$*;
(3) *$d(G) = 2$ and $Z(G) = \Phi(G)$.*

**Lemma 2.3.** *(See [7, Chapter IV, Proposition 1.5].) Let $G$ be a group having subgroups $A, B, C$, and let $N \trianglelefteq G$. If $[B,C,A] \leqslant N$ and $[C,A,B] \leqslant N$, then $[A,B,C] \leqslant N$.*

A *right operator group* is a triple $(G, \Omega, \alpha)$ consisting of a group $G$, a set $\Omega$ called the *operator domain*, and a function $\alpha : G \times \Omega \to G$ such that $g \mapsto (g,\omega)\alpha$ is an endomorphism of $G$ for each $\omega \in \Omega$. We write $g^\omega$ for $(g,\omega)\alpha$ and speak of the *$\Omega$-group $G$* if the function $\alpha$ is understood. If $G$ is an $\Omega$-group, an *$\Omega$-subgroup* of $G$ is a subgroup $H$ which is *$\Omega$-admissible*. Namely, $h^\omega \in H$ for $h \in H$ and $\omega \in \Omega$. An *$\Omega$-homomorphism* $\alpha : G \to H$ is a homomorphism between $\Omega$-groups $G$ and $H$ such that

$$\left(g^\omega\right)^\alpha = \left(g^\alpha\right)^\omega$$

for all $g \in G$ and $\omega \in \Omega$. We also speak of *$\Omega$-endomorphisms* ($\Omega$-homomorphisms from a group to itself) and *$\Omega$-automorphisms* (bijective $\Omega$-endomorphisms).

**Theorem 2.4.** *(See [7, Chapter III, Theorem 2.14].) If $G$ is a finite $\Omega$-group, then $G$ can be expressed as a direct product of finitely many non-trivial $\Omega$-subgroups*:

$$G = H_1 \times \cdots \times H_r.$$

*If $G = K_1 \times \cdots \times K_s$, where $K_j$ $(j = 1, 2, \ldots, s)$ are non-trivial $\Omega$-subgroups, then $r = s$ and there is an $\Omega$-automorphism $\alpha$ of $G$ such that, after suitable relabeling of the $K_j$'s if necessary, $H_i^{\alpha} = K_i$ and $G = K_1 \times \cdots \times K_k \times H_{k+1} \times \cdots \times H_r$ for $k = 1, 2, \ldots, r$.*

**Lemma 2.5.** *(See [7, Chapter I, Theorem 5.7].) If $K$ is a subgroup of a group $G$, then $N_G(K)/C_G(K)$ is isomorphic to a subgroup of $\mathrm{Aut}(K)$.*

**Lemma 2.6.** *If $G$ is a group of order $p^n$ where $n \geqslant 4$, then $G$ has an abelian normal subgroup of order $p^3$.*

**Proof.** Let $K$ be an abelian normal subgroup of $G$ with order $p^2$. Then Lemma 2.5 shows that $G/C_G(K)$ is isomorphic to a subgroup of $\mathrm{Aut}(K)$. It follows that $|G/C_G(K)| \leqslant p$, and hence $|C_G(K)| \geqslant p^3$. Thus there is a normal subgroup $N$ of $G$ contained in $C_G(K)$ with order $p^3$. It is easy to see that $N$ is abelian. □

## 3. Properties and examples of finite *s*-self dual *p*-groups

First, we give some properties of finite $s$-self dual $p$-groups.

**Proposition 3.1.** *Let $G$ be a finite $s$-self dual $p$-group.*

(1) *If $H \leqslant G$, then $d(H) \leqslant d(G)$.*
(2) $\exp(G) = \exp(G/G')$.
(3) *If $K$ is an abelian subgroup of $G$, then $|K| \leqslant |G/G'|$.*
(4) *If $K$ is an abelian subgroup of $G$ and $|K| = |G/G'|$, then $K \cong G/G'$.*
(5) *$G/G_n$ is $s$-self dual.*
(6) *$G/\mho_1(G')$ is $s$-self dual.*

**Proof.** (1) Since $G$ is $s$-self dual, $H$ is isomorphic to a quotient of $G$. Assume that $H \cong G/N$. Then $d(H) = d(G/N) \leqslant d(G)$.

(2) Take $a \in G$ such that $o(a) = \exp(G)$. Since $G$ is $s$-self dual, $\langle a \rangle$ is isomorphic to a quotient of $G$. Assume that $\langle a \rangle \cong G/N$. Since $G/N$ is abelian, $G' \leqslant N$. Then $G/N \cong (G/G')/(N/G')$, and hence $\exp(G/N) \leqslant \exp(G/G')$. Then $\exp(G) = o(a) = \exp(G/N) \leqslant \exp(G/G')$. On the other hand, $\exp(G/G') \leqslant \exp(G)$. Hence $\exp(G) = \exp(G/G')$.

(3) Since $G$ is $s$-self dual, $K$ is isomorphic to a quotient of $G$. Assume that $K \cong G/N$. Since $G/N$ is abelian, $G' \leqslant N$. Hence $|K| = |G/N| \leqslant |G/G'|$.

(4) If follows from the proof of (3).

(5) Let $L/G_n \leqslant G/G_n$. Then $L \leqslant G$. Since $G$ is $s$-self dual, $L$ is isomorphic to a quotient of $G$. Assume that $L \cong G/M$. Then $L/G_n$ is isomorphic to a quotient of $G/M$. Let $L/G_n \cong (G/M)/(N/M) \cong G/N$. Since $c(G/G_n) = n - 1$, we have $c(L/G_n) \leqslant n - 1$. It follows that $c(G/N) \leqslant n - 1$, and hence $G_n \leqslant N$. Then $L/G_n \cong G/N \cong (G/G_n)/(N/G_n)$. By the arbitrariness of $L/G_n$, $G/G_n$ is also $s$-self dual.

(6) Let $L/\mho_1(G') \leqslant G/\mho_1(G')$. Then $L \leqslant G$. Since $G$ is $s$-self dual, $L$ is isomorphic to a quotient of $G$. Assume that $L \cong G/M$. Then $L/\mho_1(G')$ is isomorphic to a quotient of $G/M$. Let $L/\mho_1(G') \cong (G/M)/(N/M) \cong G/N$. Since $\exp(G/\mho_1(G'))' = p$, we have $\exp(L/\mho_1(G'))' = p$. It follows that $\exp(G/N)' = p$, and hence $\mho_1(G') \leqslant N$. Then $L/\mho_1(G') \cong G/N \cong (G/\mho_1(G'))/(N/\mho_1(G'))$. By the arbitrariness of $L/\mho_1(G')$, $G/\mho_1(G')$ is also $s$-self dual. □

**Lemma 3.2.** *If $H \times \langle a \rangle$ is a finite $s$-self dual $p$-group, then every maximal subgroup of $H$ is isomorphic to a quotient of $H$.*

**Proof.** Let $G = H \times \langle a \rangle$ and let $L$ be a maximal subgroup of $H$. Then $L \times \langle a \rangle$ is also a maximal subgroup of $G$. Since $G$ is $s$-self dual, $L \times \langle a \rangle$ is isomorphic to a quotient of $G$. Assume that $L \times \langle a \rangle \cong G/M$ and $|\langle a \rangle| = p^k$.

If $M \leqslant H$, then $G/M \cong H/M \times \langle a \rangle$. Theorem 2.4 gives that $L \cong H/M$.

If $M \nleqslant H$, then $M \cap H = 1$. Let $G/M = T/M \times \langle bM \rangle$, where $T/M \cong L$, $\langle bM \rangle \cong \langle a \rangle$. Then $G = T\langle b \rangle$, $|T| = |H|$, $(bM)^{p^k} = M$ and $[b, T] \in M$. Since $|G/H| = p^k$, we deduce that $b^{p^k} \in H$. It follows from $(bM)^{p^k} = M$ that $b^{p^k} \in M$. Hence $b^{p^k} \in H \cap M = 1$. It follows that $\langle b \rangle \cong C_{p^k}$. Since $G' \leqslant H$, $[b, T] \in H \cap M = 1$. It follows that $G = T \times \langle b \rangle$. Theorem 2.4 gives that $T \cong H$. Since $L \cong T/M$, $L$ is isomorphic to a quotient of $H$. $\square$

**Lemma 3.3.** *If $H \times \langle a \rangle$ is a finite s-self dual p-group, then every subgroup of $H$ is isomorphic to a quotient of $H$. In particular, $H$ is also s-self dual.*

**Proof.** Let $L$ be a proper subgroup of $H$. We argue by induction on $|H : L|$. Let $K$ be a subgroup of $H$ such that $L$ is maximal in $K$. Since $|H : K| < |H : L|$, by induction $K$ is isomorphic to a quotient of $H$. Assume that $K \cong H/N$ and $T/N$ is a maximal subgroup of $H/N$ which is isomorphic to $L$. Then $T$ is maximal in $H$. By Lemma 3.2, $T$ is isomorphic to a quotient of $H$. Assume that $T \cong H/P$. Since $L \cong T/N$, $L$ is isomorphic to a quotient of $H/P$. Assume that $L \cong (H/P)/(M/P)$. Then $L \cong H/M$. $\square$

**Theorem 3.4.** *If $H \times M$ is a finite s-self dual p-group, where $M$ is abelian, then $H$ is also s-self dual.*

**Proof.** Since $M$ can be expressed as a direct product of cyclic groups, $H$ is $s$-self dual by Lemma 3.3. $\square$

**Definition 3.5.** A group $G$ is *basic s-self dual* if $G$ is $s$-self dual and $H' < G'$ for every proper subgroup $H$ of $G$.

**Theorem 3.6.** *If $G$ is a finite s-self dual p-group, and $H$ is a minimal element of the set $\{H \mid H' = G'\}$, then there is a subgroup $M$ of $Z(G)$ such that $G = H \times M$. Moreover, $H$ is basic s-self dual.*

**Proof.** Since $G$ is $s$-self dual, $H$ is isomorphic to a quotient of $G$. Assume that $G/M \cong H$. It follows from $G' = H' \cong (G/M)' = G'M/M \cong G'/(M \cap G')$ that $G' \cap M = 1$. Since $[M, G] \leqslant G' \cap M = 1$, $M \leqslant Z(G)$.

By the minimality of $H$, for every proper subgroup $K/M$ of $G/M$, we have $(K/M)' < (G/M)'$. Also $(HM/M)' = H'M/M \cong H'/(H' \cap M) \cong H' \cong (G/M)'$. Hence $HM/M$ is not a proper subgroup of $G/M$. It follows that $G = HM$. Since $H \cong G/M = HM/M \cong H/(H \cap M)$, we deduce that $H \cap M = 1$. Hence $G = H \times M$. Finally, Theorem 3.4 gives that $H$ is basic $s$-self dual. $\square$

We give two examples of finite non-abelian $s$-self dual $p$-groups. If $G = M_p(1,1,1) \times C_p^k$, then every subgroup of $G$ is self dual by Theorem 1.2. Hence $G$ is $s$-self dual and self dual. We prove below that if $G = M_p(n,n) \times M$, where $M$ is abelian and $\exp(M) < p^n$, then $G$ is $s$-self dual but not $q$-self dual. In fact, these two examples are the only finite non-abelian $s$-self dual $p$-groups. We start with a lemma.

**Lemma 3.7.** *Let $G = H \times M$ be a finite p-group, where $H \cong M_p(n,n)$, $M$ is abelian and* $\exp(M) < p^n$*. If $x, y \in G$ such that $[x, y] \neq 1$, then $\langle x, y \rangle \cong M_p(n,n)$ and $G = \langle x, y \rangle \times M$.*

**Proof.** Let $H = \langle a, b \mid a^{p^n} = b^{p^n} = 1,\ [a,b] = a^{p^{n-1}} \rangle$. Since $Z(G) = Z(H) \times M = \langle a^p, b^p \rangle \times M$, we deduce that $G/Z(G) = \langle aZ(G), bZ(G) \rangle$ is elementary abelian of order $p^2$. Since $[x, y] \neq 1$, we have $x, y \notin Z(G)$ and $\langle xZ(G), yZ(G) \rangle = \langle aZ(G), bZ(G) \rangle = G/Z(G)$. Hence $\langle x, y \rangle = \langle az_1, bz_2 \rangle$, where $z_1, z_2 \in Z(G)$. Without loss of generality, we let $x = az_1$, $y = bz_2$. Since $\exp(Z(G)) < p^n$, $[x, y] = [az_1, bz_2] = a^{p^{n-1}} = x^{p^{n-1}}$. Hence $\langle x, y \rangle = \langle x, y \mid x^{p^n} = y^{p^n} = 1,\ [x, y] = x^{p^{n-1}} \rangle \cong M_p(n,n)$.

Observe that $\Omega_1(\langle x, y \rangle) = \langle x^{p^{n-1}}, y^{p^{n-1}} \rangle = \langle a^{p^{n-1}}, b^{p^{n-1}} \rangle$ and $\Omega_1(\langle x, y \rangle \cap M) \leqslant \Omega_1(\langle x, y \rangle) \cap M = \langle a^{p^{n-1}}, b^{p^{n-1}} \rangle \cap M = 1$. It follows that $\langle x, y \rangle \cap M = 1$. Hence $G = \langle x, y \rangle \times M$. $\square$

**Example 3.8.** If $G = H \times M$ is a finite $p$-group, where $H \cong M_p(n,n)$, $M$ is abelian and $\exp(M) < p^n$, then $G$ is $s$-self dual but not $q$-self dual.

**Proof.** We have $Z(G) = Z(H) \times M$. It follows that $G/Z(G) \cong H/Z(H)$ is elementary abelian of order $p^2$. Since $Z(G)$ is contained in every maximal abelian subgroup of $G$, there is a bijection from the set of maximal abelian subgroups of $G$ to the set of maximal subgroups of $G/Z(G)$. Hence there are $1 + p$ maximal abelian subgroups in $G$. Let $H = \langle a, b \mid a^{p^n} = b^{p^n} = 1,\ [a, b] = a^{p^{n-1}} \rangle$. The maximal abelian subgroups of $G$ are: $\langle a^p \rangle \times \langle b \rangle \times M$ and $\langle b^p \rangle \times \langle ab^i \rangle \times M$, where $i = 0, 1, \dots, p-1$. Each is isomorphic to $G/G'$.

We now prove that every proper subgroup $L$ of $G$ is isomorphic to a quotient of $G$.

If $L$ is abelian, then there is a maximal abelian subgroup $K$ of $G$ such that $L \leqslant K$. The proof given above shows that $K \cong G/G'$. By Theorem 1.2, $K$ is self dual. It follows that $L$ is isomorphic to a quotient of $K$. Hence $L$ is isomorphic to a quotient of $G/G'$. Assume that $L \cong (G/G')/(N/G')$. Then $L \cong G/N$.

If $L$ is not abelian, then there exist $x, y \in L$ such that $[x, y] \neq 1$. By Lemma 3.7, $G = \langle x, y \rangle \times M$. Then $L = L \cap G = L \cap (\langle x, y \rangle \times M) = \langle x, y \rangle \times (L \cap M)$. Since $M$ is abelian, Theorem 1.2 gives that $M$ is self dual. It follows that $L \cap M$ is isomorphic to a quotient of $M$. Assume that $L \cap M \cong M/N$. Then $L = \langle x, y \rangle \times (L \cap M) \cong (\langle x, y \rangle \times M)/N = G/N$.

Finally we prove that $G$ is not $q$-self dual. Let $N = \langle b^p \rangle \times M$. Then $G/N \cong M_p(n, 1)$. Lemma 3.7 gives that every non-commuting pair of elements of $G$ generates $M_p(n, n)$. It follows that no subgroup of $G$ is isomorphic to $G/N$, so $G$ is not $q$-self dual. $\square$

## 4. Finite *s*-self dual *p*-groups generated by two elements

By Proposition 3.1(1), if $G$ is a finite $s$-self dual $p$-group generated by two elements, then all subgroups of $G$ are generated by at most two elements. Finite $p$-groups whose subgroups are generated by at most two elements have been classified by [2] and [8]. By checking [8, Main Theorem], we obtain the following.

**Lemma 4.1.** *Let $G$ be a finite $p$-group. If all subgroups of $G$ are generated by at most two elements, then $G$ is one of the following*:

(1) *a metacyclic $p$-group*;
(2) $M_p(1, 1, 1)$;
(3) *a 3-group of maximal class having nilpotency class greater than 2 except* $\langle a, c \mid a^9 = b^3 = c^3 = 1, [a, c] = b, [a, b] = a^3, [b, c] = 1 \rangle$;
(4) $\langle a, b, c \mid a^{p^2} = b^p = 1,\ c^p = a^{\alpha p},\ [a, c] = b,\ [a, b] = a^p,\ [b, c] = 1 \rangle$, *where $p \geqslant 5$, $\alpha = 1$ or a fixed quadratic non-residue modulo $p$.*

We start with metacyclic $p$-groups. The following lemma describes them.

**Lemma 4.2.** *(See [9, Theorem 2.1].)* (1) *Every metacyclic $p$-group $G$, for $p$ odd, is*

$$\left\langle a, b \;\middle|\; a^{p^{r+s+u}} = 1,\ b^{p^{r+s+t}} = a^{p^{r+s}},\ a^b = a^{1+p^r} \right\rangle$$

*where $r, s, t, u$ are non-negative integers with $r \geqslant 1$ and $u \leqslant r$. Different values of the parameters $r, s, t$ and $u$ give non-isomorphic metacyclic $p$-groups.*

*Furthermore, $G$ is split if and only if either $s = 0$, or $t = 0$, or $u = 0$.*

(2) *Let $G$ be a metacyclic 2-group. Then $G$ is one of the following.*

(I) $G = \langle a, b\colon\ a^{2^{v+t'+u+1}} = 1,\ b^{2^{t+1}} = a^{2^{v+t'+1}},\ a^b = a^{-1+2^{v+u+1}} \rangle$, *where $v, t, t'$ and $u$ are non-negative integers with $u \leqslant 1$, $t' \leqslant 1$, $tt' = tv = ut' = 0$ and if $t + u + v = 0$, then $t' = 0$.*
(II) *Ordinary metacyclic 2-group*: $G = \langle a, b\colon\ a^{2^{r+s+u}} = 1,\ b^{2^{r+s+t}} = a^{2^{r+s}},\ a^b = a^{1+2^r} \rangle$, *where $r, s, t, u$ are non-negative integers with $r \geqslant 2$ and $u \leqslant r$.*

(III) *Exceptional metacyclic 2-group*: $G = \langle a, b\colon a^{2^{r+s+v+t'+u}} = 1,\ b^{2^{r+s+t}} = a^{2^{r+s+v+t'}},\ a^b = a^{-1+2^{r+v}}\rangle$, *where* $r, s, v, t, t', u$ *are non-negative integers with* $r \geqslant 2,\ t' \leqslant r,\ u \leqslant 1,\ tt' = sv = tv = 0$, *and if* $t' \geqslant r-1$ *then* $u = 0$.

*Groups of different types, or of the same type but with different values of parameters, are not isomorphic to each other.*

*A Type* (I) *group* $G \neq C_2 \times C_2$ *is split if and only if* $(t, u) \neq (0, 1)$; *a Type* (II) *group is split if and only if either* $s = 0$, *or* $t = 0$, *or* $u = 0$; *a Type* (III) *group is split if and only if* $u = 0$.

**Lemma 4.3.** *If G is a metacyclic p-group, then G is s-self dual if and only if G is abelian or* $G \cong M_p(n, n)$.

**Proof.** If $G$ is abelian or $G \cong M_p(n, n)$, then $G$ is $s$-self dual by Theorem 1.2 and Example 3.8. We only need to prove the necessity.

(1) Assume $p$ is odd. By Lemma 4.2, we may let $G = \langle a, b \mid a^{p^{r+s+u}} = 1,\ b^{p^{r+s+t}} = a^{p^{r+s}},\ b^{-1}ab = a^{1+p^r}\rangle$, where $r$, $s$, $t$, $u$ are non-negative integers with $r \geqslant 1$ and $u \leqslant r$.

Let $H_1 = \langle a^{p^{s+u}}, b\rangle$ and $H_2 = \langle a^{p^{s+u-1}}, b^p\rangle$. By calculation, $H_1$ and $H_2$ are abelian. Since $G/G'$ has type invariant $(p^{r+s+t}, p^r)$ and $|H_1| = |H_2| = |G/G'| = p^{2r+s+t}$, Proposition 3.1 gives that $H_1 \cong H_2 \cong G/G'$. It follows that $\exp(H_1) = \exp(H_2) = \exp(G/G') = p^{r+s+t}$. Since $\exp(H_1) = p^{r+s+t+u}$ and $\exp(H_2) = \max\{p^{r+1}, p^{r+s+t+u-1}\}$, we have $r+s+t+u = r+1$ and $u = 0$. Hence $s + t = 1$.

If $s = 0$ and $t = 1$, then $G' = 1$, and hence $G$ is abelian. If $s = 1$ and $t = 0$, then $|G'| = p$. In this case $G = \langle a, b \mid a^{p^{r+1}} = 1, b^{p^{r+1}} = 1,\ b^{-1}ab = a^{1+p^r}\rangle \cong M_p(r+1, r+1)$.

(2) Assume $p = 2$. If $G$ is a group of Type (I) in Lemma 4.2, then $G$ has cyclic maximal subgroup. It is easy to see that $G$ is abelian.

If $G$ is a group of Type (II) in Lemma 4.2, then $G \cong M_2(n, n)$ by a method similar to (1).

If $G$ is a group of Type (III) in Lemma 4.2, then we may let $G = \langle a, b \mid a^{2^{r+s+v+t'+u}} = 1,\ b^{2^{r+s+t}} = a^{2^{r+s+v+t'}},\ a^b = a^{-1+2^{r+v}}\rangle$, where $r$, $s$, $v$, $t$, $t'$, $u$ are non-negative integers with $r \geqslant 2$, $t' \leqslant r$, $u \leqslant 1$, $tt' = sv = tv = 0$, and if $t' \geqslant r-1$ then $u = 0$. Let $H_1 = \langle a^{2^{r+s+v+t'+u-1}}, b\rangle$ and $H_2 = \langle a^{2^{r+s+v+t'+u-2}}, b^2\rangle$. By calculation, $H_1$ and $H_2$ are abelian and $|H_1| = |H_2| = |G/G'| = 2^{r+s+t+1}$. It follows from Proposition 3.1 that $H_1 \cong H_2$, and hence $\exp(H_1) = \exp(H_2)$. Since $\exp(H_1) = 2^{r+s+t+u}$ and $\exp(H_2) = \max\{2^2, 2^{r+s+t+u-1}\}$, we have $r+s+t+u = 2$. Since $r \geqslant 2$ and $r$, $s$, $t$, $u$ are non-negative integers, we have $r = 2$, $s = t = u = 0$. In this case $G = \langle a, b \mid a^{2^{r+v+t'}} = 1,\ b^{2^r} = 1,\ a^b = a^{-1+2^{r+v}}\rangle$. It follows from Proposition 3.1(2) that $t' + v = 0$. Thus $G = \langle a, b \mid a^4 = b^4 = 1, [a, b] = a^2\rangle \cong M_2(2, 2)$. □

**Theorem 4.4.** *If G is a finite non-abelian p-group generated by two elements, then G is s-self dual if and only if G is isomorphic to* $M_p(1, 1, 1)$ *or* $M_p(n, n)$.

**Proof.** We only need to check those groups listed in Lemma 4.1.

(1) If $G$ is metacyclic, then it follows from Lemma 4.3 that $G \cong M_p(n, n)$.

(2) If $G \cong M_p(1, 1, 1)$, then $G$ is $s$-self dual.

(3) If $G$ is a 3-group of maximal class having nilpotency class greater than 2, then $|G| \geqslant 3^4$. It follows from Lemma 2.6 that $G$ has an abelian normal subgroup of order $p^3$. Since $|G/G'| = p^2$, Proposition 3.1(3) gives that $G$ is not $s$-self dual.

(4) $G = \langle a, b, c \mid a^{p^2} = b^p = 1, c^p = a^{\alpha p}, [a, c] = b, [a, b] = a^p, [b, c] = 1\rangle$, where $p \geqslant 5$, $\alpha = 1$ or a fixed quadratic non-residue modulo $p$. In this case, $G$ has an abelian maximal subgroup $H = \langle a^p\rangle \times \langle b\rangle \times \langle c\rangle$, and $|G/G'| = p^2$. It follows from Proposition 3.1(3) that $G$ is not $s$-self dual. □

## 5. The lower central series of a finite *p*-group

First, we need a theorem about symplectic bilinear mappings.

**Definition 5.1.** Let $V$, $W$ be finite-dimensional vector spaces over a field $\mathbb{F}$ and let $f$ be a symplectic bilinear mapping of $V \times V$ into $W$. For $v \in V$, define

$$v^{\perp} = \{u \mid u \in V,\ f(u, v) = 0\}.$$

Then $v^{\perp}$ is an $\mathbb{F}$-subspace of $V$, and since $f$ is symplectic, $\mathbb{F}v \subseteq v^{\perp}$.

**Theorem 5.2.** *(See [4, Chapter VIII, Theorem 9.8].) Let $V$, $W$ be finite-dimensional vector spaces over a field $\mathbb{F}$ and let $f$ be a symplectic bilinear mapping of $V \times V$ into $W$. Suppose that $f(X, X) \subset f(V, V)$ for every proper subspace $X$ of $V$. Then $V$ is spanned by*

$$\{v \mid v \in V,\ v^{\perp} = \mathbb{F}v\}.$$

**Lemma 5.3.** *Let $V$, $W$ be finite-dimensional vector spaces over a field $\mathbb{F}$ and let $f$ be a symplectic bilinear mapping of $V \times V$ into $W$. Suppose that the dimension of $f(V, V)$ is $k$.*

(1) *There is a subspace $S$ of $V$ such that $f(S, S) = f(V, V)$ and the dimension of $S$ does not exceed $k + 1$.*
(2) *Let $k \geqslant 2$ and let $S$ be a $(k + 1)$-dimensional subspace of $V$ such that $f(S, S) = f(V, V)$. If $f(X, X) \subset f(S, S) = f(V, V)$ for every proper subspace $X$ of $S$, then $S$ has a 2-dimensional subspace $A$ such that $f(A, A) = 0$.*

**Proof.** (1) Let $S$ be a subspace of $V$ such that $f(S, S) = f(V, V)$ and $f(X, X) \subset f(S, S) = f(V, V)$ for every proper subspace $X$ of $S$. It follows from Theorem 5.2 that $S$ is spanned by

$$\{s \mid s \in S,\ s^{\perp} = \mathbb{F}s\}.$$

Let $s_1, s_2, \ldots, s_d$ be a basis of $S$, where $s_i^{\perp} = \mathbb{F}s_i$ $(i = 1, 2, \ldots, d)$. We only need to prove that $d \leqslant k + 1$. Otherwise, $d \geqslant k + 2$. Since the dimension of $f(S, S) = f(V, V)$ is $k$, we deduce that $f(s_1, s_2), f(s_1, s_3), \ldots, f(s_1, s_d)$ are linearly dependent. Hence there exists a non-trivial linear combination $t = j_2 s_2 + j_3 s_3 + \cdots + j_d s_d$ such that $f(s_1, t) = 0$, contrary to $s_1^{\perp} = \mathbb{F}s_1$.

(2) Assume the contrary. We claim that if $a_1, a_2, \ldots, a_{k+1}$ is a basis of $S$, then $f(a_1, a_2), f(a_1, a_3), \ldots, f(a_1, a_{k+1})$ are linearly independent. (Otherwise, consider a non-trivial linear combination $\beta = j_2 a_2 + j_3 a_3 + \cdots + j_{k+1} a_{k+1}$ such that $j_2 f(a_1, a_2) + j_3 f(a_1, a_3) + \cdots + j_{k+1} f(a_1, a_{k+1}) = 0$. Let $A$ be the subspace generated by $a_1$ and $\beta$. Then $A$ is a 2-dimensional subspace of $S$ such that $f(A \times A) = 0$, a contradiction.)

Then $f(S, S) = L(f(a_1, a_2), f(a_1, a_3), \ldots, f(a_1, a_k)) \oplus L(f(a_1, a_{k+1}))$. Similarly, $f(S, S) = L(f(a_2, a_1), f(a_2, a_3), \ldots, f(a_2, a_k)) \oplus L(f(a_2, a_{k+1}))$.

Let $M = L(f(a_1, a_2), f(a_1, a_3), \ldots, f(a_1, a_k))$ and $N = L(f(a_2, a_1), f(a_2, a_3), \ldots, f(a_2, a_k))$. If $M \neq N$, letting $X = L(a_1, a_2, \ldots, a_k)$, then $f(X, X) = f(S, S)$, contrary to the hypothesis. Hence we have $M = N$.

Let $f(a_2, a_{k+1}) = i f(a_1, a_{k+1}) + m$ where $m \in M$. Then $f(a_2 - i a_1, a_{k+1}) = m \in M$. Let $a_2' = a_2 - i a_1$ and $N' = L(f(a_2', a_1), f(a_2', a_3), \ldots, f(a_2', a_k))$. Then $M = L(f(a_1, a_2'), f(a_1, a_3), \ldots, f(a_1, a_k))$ and $N' = M$. Since $f(a_2', a_{k+1}) \in M = N'$, $f(a_2', a_1), f(a_2', a_3), \ldots, f(a_2', a_{k+1})$ are linearly dependent, contrary to the above assertion. $\square$

**Lemma 5.4.** *If $G$ is a finite $p$-group, $c(G) = 2$ and $G' \cong C_p^k$, then $G$ has a subgroup $K$ such that $K' = G'$ and $d(K) \leqslant k + 1$.*

**Proof.** Since $G' \cong C_p^k$, it may be viewed as a $k$-dimensional linear space over $GF(p)$. Let $\overline{G} = G/\Phi(G)$. Then $\overline{G}$ may be considered as a linear space over $GF(p)$. Let $f$ be a mapping from $\overline{G} \times \overline{G}$ to $G'$ such that $f(\overline{g}, \overline{h}) = [g, h]$ for all $\overline{g}, \overline{h} \in \overline{G}$. It follows from $c(G) = 2$ that $f$ is a symplectic bilinear mapping and $f(\overline{G}, \overline{G}) = G'$. By Lemma 5.3(1), there is a subspace $\overline{K}$ of $\overline{G}$ such that $f(\overline{K}, \overline{K}) = G'$ and the dimension of $\overline{K}$ does not exceed $k + 1$. Assume that $\overline{K} = L(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_s})$, where $s \leqslant k + 1$. If $K = \langle x_1, x_2, \ldots, x_s \rangle$, then $K' = G'$, and $d(K) \leqslant k + 1$. $\square$

**Lemma 5.5.** *If $G$ is a finite $p$-group, $c(G) = 2$ and $d(G') = k$, then $G$ has a subgroup $K$ such that $K' = G'$ and $d(K) \leqslant k + 1$.*

**Proof.** Let $\overline{G} = G/\mho_1(G')$. Then $\overline{G}' \cong C_p^k$. It follows from Lemma 5.4 that $\overline{G}$ has a subgroup $\overline{K}$ such that $\overline{K}' = \overline{G}'$ and $d(\overline{K}) \leqslant k+1$. Assume that $\overline{K} = \langle \overline{a_1}, \overline{a_2}, \dots, \overline{a_s}\rangle$ where $s \leqslant k+1$, and let $K = \langle a_1, a_2, \dots, a_s\rangle$. Then $K'\mho_1(G') = G'$. Hence $K' = G'$ and $d(K) \leqslant k + 1$. $\square$

**Theorem 5.6.** *If $G$ is a finite non-abelian $p$-group and $|G'| = p^k$, then $G$ has a subgroup $K$ such that $d(K) \leqslant k + 1$ and $K_n = G_n$ for all $2 \leqslant n \leqslant c(G)$.*

**Proof.** We argue by induction on $c = c(G)$. If $c = 2$, then the result follows from Lemma 5.5. We assume that $c \geqslant 3$. Suppose that $|G_c| = p^s$, and let $\overline{G} = G/G_c$. Then $c(\overline{G}) = c - 1$ and $|\overline{G}'| = p^{k-s}$. By induction, $\overline{G}$ has a subgroup $\overline{H}$ such that $d(\overline{H}) \leqslant k - s + 1$ and $\overline{H}_n = \overline{G}_n$ for all $2 \leqslant n \leqslant c - 1$. Assume that $\overline{H} = \langle \overline{a_1}, \overline{a_2}, \dots, \overline{a_t}\rangle$ where $t \leqslant k - s + 1$, and let $H = \langle a_1, a_2, \dots, a_t\rangle$. Then $H_n G_c = G_n$ for all $2 \leqslant n \leqslant c$.

First, we assume that $\Phi(G_c) = 1$, namely, $G_c$ is elementary abelian. Since $G_c = [G_{c-1}, G] = [H_{c-1}G_c, G] = [H_{c-1}, G]$, there exist $h_i \in H_{c-1}$ and $g_i \in G$ such that $G_c = \langle [h_i, g_i] \mid i = 1, 2, \dots, s\rangle$. Let $K = \langle H, g_1, g_2, \dots, g_s\rangle$. Then $K_c \geqslant [H_{c-1}, K] \geqslant G_c$. Obviously, $K_n \geqslant H_n$ and $K_n \geqslant K_c = G_c$ for all $2 \leqslant n \leqslant c$. It follows that $K_n \geqslant H_n G_c = G_n$ and hence $K_n = G_n$.

Now assume that $|\Phi(G_c)| = p^u$ where $u \geqslant 1$, and let $\tilde{G} = G/\Phi(G_c)$. Then $\tilde{G}'$ is elementary abelian and $|\tilde{G}'| = p^{k-u}$. By the case of $\Phi(G_c) = 1$, $\tilde{G}$ has a subgroup $\tilde{K}$ such that $d(\tilde{K}) \leqslant k-u+1$ and $\tilde{K}_n = \tilde{G}_n$ for all $2 \leqslant n \leqslant c$. Assume that $\tilde{K} = \langle \tilde{b_1}, \tilde{b_2}, \dots, \tilde{b_v}\rangle$ where $v \leqslant k - u + 1$, and let $K = \langle b_1, b_2, \dots, b_v\rangle$. Then $K_n\Phi(G_c) = G_n$ for all $2 \leqslant n \leqslant c$. Since $\Phi(G_c) \leqslant \Phi(G_n)$, we deduce that $K_n = G_n$ for all $2 \leqslant n \leqslant c$. $\square$

## 6. Finite *s*-self dual *p*-groups with elementary abelian derived group

**Lemma 6.1.** *If $G$ is a basic finite $s$-self dual $p$-group, $c(G) = 2$ and* $\exp(G') = p$, *then $|G'| = p$.*

**Proof.** Assume that $|G'| = p^k$. Then $d(G) \leqslant k + 1$ by Theorem 5.6 and Definition 3.5. Since $c(G) = 2$, $G' \leqslant Z(G)$ and $G' \cong C_p^k$. It follows from Proposition 3.1(1) that $d(G) \geqslant d(G') = k$. Hence $d(G) = k$ or $k + 1$.

If $d(G) = k$, letting $a$ be an element of $G$ with maximal order, then $\langle a\rangle$ is isomorphic to a quotient of $G$ since $G$ is $s$-self dual. Assume that $\langle a\rangle \cong G/N = \langle bN\rangle$. Then $b$ is also an element of $G$ with maximal order and $\langle b\rangle \cap N = 1$. Since $G' \leqslant N$, we have $\langle b\rangle \cap G' = 1$. Hence $\langle b, G'\rangle = G' \times \langle b\rangle$ is an abelian group generated by $k + 1$ elements, which is contradicted by Proposition 3.1(1). Hence $d(G) = k + 1$.

To prove $|G'| = p$, we only need to prove $k = 1$. Assume that $k \geqslant 2$. Since $G' \cong C_p^k$, it may be viewed as a $k$-dimensional linear space over $GF(p)$. Let $\overline{G} = G/\Phi(G)$. Then $\overline{G}$ may be viewed as a $(k + 1)$-dimensional linear space over $GF(p)$. Let $f$ be a mapping from $\overline{G} \times \overline{G}$ to $G'$ such that $f(\bar{g}, \bar{h}) = [g, h]$ for all $\bar{g}, \bar{h} \in \overline{G}$. It follows from $c(G) = 2$ that $f$ is a symplectic bilinear mapping and $f(\overline{G}, \overline{G}) = G'$. Since $G$ is basic $s$-self dual, $f(\overline{X}, \overline{X}) \subset G'$ for every proper subspace $\overline{X}$ of $\overline{G}$. By Lemma 5.3(2), $\overline{G}$ has a 2-dimensional subspace $\overline{A}$ such that $f(\overline{A}, \overline{A}) = 0$. Assume that $\overline{A} = L(\overline{x_1}, \overline{x_2})$ and let $A = \langle x_1, x_2, \Phi(G)\rangle$. Then $|G/A| = p^{k-1}$, and hence $|A| = \frac{|G|}{p^{k-1}}$. Since $c(G) = 2$ and $\exp(G') = p$, we have $\Phi(G) \leqslant Z(G)$, and hence $A$ is abelian. Moreover, $|A| = \frac{|G|}{p^{k-1}} > \frac{|G|}{p^k} = |G/G'|$, which is contradicted by Proposition 3.1(3). $\square$

**Corollary 6.2.** *If $G$ is a finite $s$-self dual $p$-group, $c(G) = 2$ and* $\exp(G') = p$, *then $|G'| = p$.*

**Proof.** By Theorem 3.6, $G$ is a direct product of $H$ and $M$, where $H$ is basic $s$-self dual and $M$ is abelian. Lemma 6.1 shows that $|H'| = p$. Hence $|G'| = p$. $\square$

**Lemma 6.3.** *If $G$ is a finite $s$-self dual $p$-group and* $\exp(G') = p$*, then* $|G'| = p$.

**Proof.** By Corollary 6.2, we only need to prove $c(G) = 2$. Otherwise, let $G$ be a counter-example with minimal order. It follows from Theorem 3.6 that $G$ is a basic finite $s$-self dual $p$-group. By Proposition 3.1(5) and the minimality of $G$, $c(G) = 3$.

Let $\overline{G} = G/G_3$. Then $c(\overline{G}) = 2$. By Proposition 3.1(5), $\overline{G}$ is $s$-self dual. Since $\exp(G') = p$, $\exp(\overline{G}') = p$. Then $|\overline{G}'| = p$ by Corollary 6.2. Assume that $\overline{G}' = \langle[\bar{a}, \bar{b}]\rangle$. Then we may let $\overline{G} = \langle\bar{a}, \bar{b}\rangle \times \langle\bar{c}_1\rangle \times \cdots \times \langle\bar{c}_t\rangle$ by Theorem 3.6. Moreover, we may let $G = \langle a, b, c_1, c_2, \ldots, c_t\rangle$. Since $|G'/G_3| = |\overline{G}'| = p$, we have $G' = \langle[a, b], G_3\rangle$. Since $[a, c_i] \in G_3 \leqslant Z(G)$, $[a, c_i, b] = 1$. This also implies that $[c_i, b, a] = 1$. By Lemma 2.3, $[b, a, c_i] = 1$. Hence $G_3 = [G', G] = \langle[b, a, c_i],\ [b, a, a],\ [b, a, b]\rangle = \langle[b, a, a],\ [b, a, b]\rangle$. If $S = \langle a, b\rangle$, then $S' = G'$. Since $G$ is basic $s$-self dual, $G = S = \langle a, b\rangle$. By Theorem 4.4, $G$ is $M_p(1, 1, 1)$ or $M_p(n, n)$, which contradicts $c(G) = 3$. $\square$

The following theorem describes the structure of a finite $s$-self dual $p$-group whose derived group is elementary abelian.

**Theorem 6.4.** *If $G$ is a finite $s$-self dual $p$-group with elementary abelian derived group, then $G$ is isomorphic to one of the following*:

(1) $M_p(1, 1, 1) \times M$*, where $M$ is abelian and* $\exp(M) \leqslant p$;
(2) $M_p(n, n) \times M$*, where $M$ is abelian and* $\exp(M) < p^n$.

**Proof.** By Lemma 6.3, $|G'| = p$. Let $H$ be a minimal non-abelian subgroup of $G$. Then $H$ is a minimal element of the set $\{H \mid H' = G'\}$. By Theorem 3.6, there is a subgroup $M$ of $Z(G)$ such that $G = H \times M$ and $H$ is basic $s$-self dual. It follows from Theorem 4.4 that $H$ is isomorphic to $M_p(1, 1, 1)$ or $M_p(n, n)$.

(1) If $H$ is isomorphic to $M_p(1, 1, 1)$, then $G \cong M_p(1, 1, 1) \times M$. In this case, we claim that $G$ is $s$-self dual if and only if $\exp(M) \leqslant p$.

Sufficiency follows from Theorem 1.2. We only need to prove the necessity. Take $d \in M$ such that $o(d) = \exp(M)$, and let $A = \langle ad, b\rangle$. Then $A$ is minimal non-abelian by Theorem 2.2. Hence $A$ is a minimal element of the set $\{H \mid H' = G'\}$. By Theorem 3.6, there is a subgroup $L$ of $Z(G)$ such that $G = A \times L$ and $A$ is basic $s$-self dual. Since $A$ cannot be isomorphic to $M_p(n, n)$, we have $A \cong M_p(1, 1, 1)$. Hence $\exp(M) = o(d) \leqslant p$.

(2) If $H$ is isomorphic to $M_p(n, n)$, then $G \cong M_p(n, n) \times M$. In this case, we claim that $G$ is $s$-self dual if and only if $\exp(M) < p^n$.

Sufficiency follows from Example 3.8. We only need to prove the necessity. Take $c \in M$ such that $o(c) = \exp(M)$, and let $K = \langle ac, b\rangle$. Then $K$ is minimal non-abelian by Theorem 2.2. Hence $K$ is a minimal element of the set $\{H \mid H' = G'\}$. By Theorem 3.6, there is a subgroup $L$ of $Z(G)$ such that $G = K \times L$ and $K$ is basic $s$-self dual. Since $K$ cannot be isomorphic to $M_p(1, 1, 1)$, we have $K \cong M_p(n, n)$. Hence $\exp(M) = o(c) \leqslant p^n$. If $\exp(M) = p^n$, then $K = \langle ac, b \mid (ac)^{p^n} = b^{p^n} = 1, [ac, b] = a^{p^{n-1}}\rangle \ncong M_p(n, n)$. Hence we have $\exp(M) < p^n$. $\square$

## 7. Finite $s$-self dual groups and self dual groups

Now we determine the structure of finite $s$-self dual $p$-groups.

**Theorem 7.1.** *If $G$ is a finite $s$-self dual $p$-group, then $G$ is isomorphic to one of following*:

(1) *an abelian $p$-group*;
(2) $M_p(1, 1, 1) \times M$*, where $M$ is abelian and* $\exp(M) \leqslant p$;
(3) $M_p(n, n) \times M$*, where $M$ is abelian and* $\exp(M) < p^n$.

**Proof.** Assume $G$ is not abelian and let $H$ be a minimal element of the set $\{H \mid H' = G'\}$. By Theorem 3.6, there is a subgroup $M$ of $Z(G)$ such that $G = H \times M$ and $H$ is basic $s$-self dual. It follows from

Proposition 3.1(6) that $H/\mho_1(H')$ is also $s$-self dual. Since $\exp(H/\mho_1(H'))' = \exp(H'/\mho_1(H')) = p$, Theorem 6.4 shows that $|H/\mho_1(H'))'| = |H'/\mho_1(H')| = p$. Hence $H'$ is cyclic. Moreover, there exist $a, b \in H$ such that $\langle [a,b]\rangle = H'$. The minimality of $H$ implies that $H = \langle a, b\rangle$. Theorem 4.4 shows that $|H'| = p$. Hence $G$ is (2) or (3) by Theorem 6.4. □

**Corollary 7.2.** *If $P$ is a finite $p$-group, then $P$ is self dual if and only if $P$ is abelian or $P \cong M_p(1,1,1) \times M$, where $M$ is abelian and $\exp(M) \leqslant p$.*

**Proof.** It follows from Theorem 7.1 and Example 3.8. □

**Theorem 7.3.** *A finite group $G$ is $s$-self dual if and only if $G$ is nilpotent and all Sylow subgroups of $G$ are $s$-self dual.*

**Proof.** Let ${p_1}^{\alpha_1} {p_2}^{\alpha_2} \cdots {p_s}^{\alpha_s}$ be the standard factorization of $|G|$.

*Necessity*: Let $P_i \in \mathrm{Syl}_{p_i}(G)$. Since $G$ is $s$-self dual, $P_i$ is isomorphic to a quotient of $G$. It follows that $G$ is $p_i$-nilpotent. Hence $G$ is nilpotent. Suppose that $H_i$ is a subgroup of $P_i$. Since $G$ is $s$-self dual, $H_i$ is isomorphic to a quotient of $G$. Assume that $H_i \cong G/N_i$. Since $G/N_i$ is a $p_i$-subgroup, $G = P_i N_i$. It follows that $H_i \cong P_i N_i/N_i \cong P_i/P_i \cap N_i$. Hence $P_i$ is also $s$-self dual.

*Sufficiency*: Let $G = P_1 \times P_2 \times \cdots \times P_s$, where $P_i \in \mathrm{Syl}_{p_i}(G)$. If $H$ is a subgroup of $G$, then $H = H_1 \times H_2 \times \cdots \times H_s$, where $H_i \leqslant P_i$, $i = 1, 2, \ldots, s$. Since $P_i$ is $s$-self dual, there exist $N_i \trianglelefteq P_i$ such that $P_i/N_i \cong H_i$. Then $G/(N_1 \times N_2 \times \cdots \times N_s) = (P_1 \times P_2 \times \cdots \times P_s)/(N_1 \times N_2 \times \cdots \times N_s) \cong P_1/N_1 \times P_2/N_2 \times \cdots \times P_s/N_s \cong H_1 \times H_2 \times \cdots \times H_s = H$. Hence $G$ is also $s$-self dual. □

By Theorems 7.1 and 7.3, the structure of finite $s$-self dual groups is completely determined. By Theorem 1.1 and Corollary 7.2, the structure of finite self dual groups is completely determined.

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## References

[1] Y. Berkovich, Z. Janko, Groups of Prime Power Order II, Walter de Gruyter, Berlin, New York, 2008.
[2] N. Blackburn, Generalization of certain elementary theorems on $p$-groups, Proc. Lond. Math. Soc. (3) 11 (1961) 1–22.
[3] B. Huppert, Endliche Gruppen I, Springer-Verlag, 1967.
[4] B. Huppert, N. Blackburn, Finite Group II, Springer-Verlag, 1982.
[5] L. Rédei, Das schiefe Produkt in der Gruppentheorie, Comment. Math. Helv. 20 (1947) 225–267.
[6] A.E. Spencer, Self dual finite groups, Ist. Venneto Sci. Lett. Arti Atti Cl. Sci. Mat. Natur. 130 (1971/72) 385–391, MR0322052 (48 #416).
[7] Mingyao Xu, An Introduction to Finite Groups, Science Press, 2001 (in Chinese).
[8] Mingyao Xu, Lijian An, Qinhai Zhang, Finite $p$-groups all of whose non-abelian proper subgroups are generated by two elements, J. Algebra 319 (2008) 3603–3620.
[9] Mingyao Xu, Qinhai Zhang, A classification of metacyclic 2-groups, Algebra Colloq. 13 (2006) 25–34.